LUÍSA MADUREIRA

Problems of Advanced Engineering Mathematics

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

$$\int_C f \cdot d\alpha \qquad \mathcal{L}\{f(t)\}$$

$$y' + P(x)y = Q(x)$$

$$\iiint_V divF dV$$

engebook

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CONTENTS

INTRODUCTION

This book is designed to assist undergraduate students in developing a good background in Engineering Mathematics. It follows the structure and topics typically found in Mathematical Analysis courses for engineers. It may also serve to graduate students, for they often look for references when they need to apply their mathematical knowledge. For instructors, it provides a practical teaching resource, offering a concise synthesis of materials and exercises suitable for a one-semester course.

Problems of Advanced Engineering Mathematics is a natural continuation of the author's two previous Portuguese editions, *Problemas de Equações Diferenciais Ordinárias e Transformadas de Laplace* and *Problemas de Análise Matemática para Engenharia*. In this volume, the author presents a revised and unified version of the two books, including the most important chapters and a few updates. The new English edition can now be used by international and mobility students and aims to reach engineering learners around the world.

Each chapter includes selected worked examples. Step by step, students can use the fundamental skills to solve problems in engineering mathematics.

Throughout the book, 450 problems are proposed, each accompanied by the correct answers so students can check their homework.

The main topics of fundamental engineering mathematics are covered: differential equations in Chapters 1 through 3, Laplace transforms in Chapter 4, line integrals in Chapter 5, surface integrals and Gauss' and Stokes' theorems in Chapter 6 and finally, Chapter 7 focuses on Fourier series and presents the solution of the vibrating string equation using the method of separation of variables and Fourier series.

CHAPTER 1

FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

In this chapter, first-order differential equations containing the first derivative of the function y are studied, which are represented by Equation 1. The independent variable is usually x, and so the function $y(x)$ is the dependent variable. All derivatives are ordinary derivatives of $y(x)$ with respect to a single independent variable x.

$$F(x, y, y') = 0 \tag{1}$$

First-order equations are too general and it is not possible to obtain a method to solve them all. Throughout the chapter integration processes are established for certain types of differential equations. The first equations considered have the derivative explicitly given as $y' = f(x, y)$. Section 1.6 deals with some examples of differential equations where $y'(x)$ is defined implicitly by $y = f(x, y')$.

Definition: **order of a differential equation**

Order of a differential equation is the order of the highest-order derivative of the dependent variable which appears in the equation.

1.1 Separable differential equations

A separable differential equation is an equation that takes the form

$$g(y)y' = f(x) \tag{2}$$

the perpendicularity relationship between tangent lines to these curves at the point of intersection is given by the equation

$$y_2' = -\frac{1}{y_1'}$$

Orthogonal trajectories of the family $F(x, y)$ are then obtained from the differential equation

$$F^*\left(x, y, -\frac{1}{y'}\right) = 0 \qquad (9)$$

and its integration giving $y_{ort}(x)$.
Equation 9 is a first-order differential equation that can be solved using the methods for determining the solution of first-order differential equations described throughout this chapter.

Example 1.9

Determine the equation of the family of curves orthogonal to the one-parameter family $x^2 + y^2 = 2ax$.

Answer

The given family is a family of circles with centers on the x-axis and tangent to the y-axis (Figure 1.1).

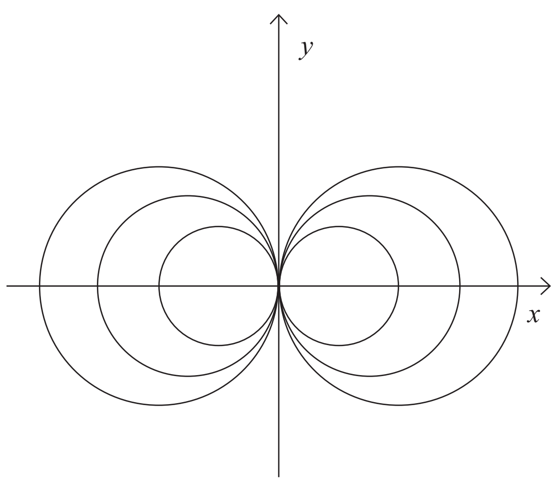


Figure 1.1 Representation of trajectories

$$x^2 + \frac{y^2}{2} = C$$

which is a family of ellipses (Figure 1.3).

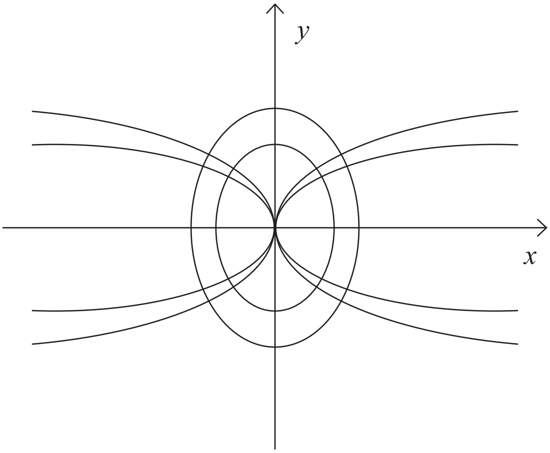


Figure 1.3 Representation of orthogonal trajectories

Problems

Determine the equation of the orthogonal trajectories of the following families of curves:

1.30 $y = ae^{x^2}, a > 0$
1.31 $y = acosx$
1.32 $y^2 + 2ax = 0, \quad a > 0$
1.33 $y = ax^n$
1.34 $x^k + y^k = a^k$
1.35 $y^2 + x^2 = ax^4$
1.36 $cos\, x\, coshy = a$
1.37 $xy^2 - 4ax^2 = 0$
1.38 $x = ae^{-y^2}$
1.39 $y = axe^x$

Answers

1.30 $y^2 = C - lnx$
1.31 $y^2 = ln(Csin^2x)$
1.32 $2x^2 + y^2 = C$
1.33 $x^2 + ny^2 = C$

CHAPTER 2

HIGHER-ORDER ORDINARY DIFFERENTIAL EQUATIONS

In this chapter, ordinary differential equations of order n usually represented by

$$F\left(x, y, y', y'', \cdots, y^{(n)}\right) = 0 \qquad (1)$$

are studied. Methods for solving these equations are only a few specific ones, and for linear equations a theory is presented.

2.1 Reducing the order of differential equations

In some particular cases it is possible to reduce the order of a differential equation like Equation 1 obtaining differential equations of lower order that are simpler to solve.
Two cases in which this reduction of order applies are approached next.

i) The equation does not contain the dependent variable y and some of its derivatives $y', \ldots y^{(k-1)}$ explicitly:

$$F\left(x, y^{(k)}, y^{(k+1)}, \cdots, y^{(n)}\right) = 0$$

With a change of variable like

$$z = y^{(k)}$$

the order of the equation is reduced in k units by obtaining

$$F\left(x, z, z', \cdots, z^{(n-k)}\right) = 0$$

2.10 $y = \frac{1}{2}xe^{x} - \frac{3}{4}e^{x} - C_1e^{-x} + C_2$
2.11 $l = 41cm, \quad v = 6cms^{-1}$

2.2 Linear differential equations of order n

Linear differential equations of order n are of the type

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x) \qquad (2)$$

with $a_0(x) \neq 0$.

This equation is linear in y and its derivatives. The coefficients $a_0(x), a_1(x), \dots, a_n(x)$ and $f(x)$ are continuous functions in a domain of validity. In the case of $f(x) \equiv 0$ the equation is said to be the associated homogeneous equation and is written as

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \qquad (3)$$

Theorem
Consider Equation 2 under the above conditions. If x_0 is a point of the domain of $a_0(x), a_1(x), \dots, a_n(x), f(x)$ and given real values $k_0, k_1, \dots, k_{n-1}$, then, there exists one only solution $y(x)$ of Equation 2 such that

$$y(x_0) = k_0, y'(x_0) = k_1, \dots y^{(n-1)}(x_0) = k_{n-1}.$$

2.2.1 Solutions of the homogeneous and non-homogeneous equation. Main theorems

Theorem
If the functions $y_1, y_2, \dots, y_m$ are m particular solutions of Equation 3 then, any linear combination of these solutions is also a solution of the homogeneous equation.

CHAPTER 3

SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

The formulation of the problem of a physical system´s behavior with n degrees of freedom, leads to a system of n simultaneous differential equations in which the independent variable is usually time.

In this chapter, only linear systems in the dependent variable consisting of n differential equations involving n unknowns $x_1, x_2, \dots, x_n$ will be considered

$$\begin{cases} x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\ x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\ \qquad \vdots \\ x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t) \end{cases} \tag{1}$$

Theorem (existence and uniqueness of the solution)

If the coefficients $a_{ij}(t)$ of this system and $f_i(t)$ are all continuous functions over an interval I, if $t_0 \in I$ and if $k_1, k_2, \dots, k_n$ are n arbitrary constants, then there exists one and only one solution $x_1(t), x_2(t), \dots, x_n(t)$ in I such that $x_1(t_0) = k_1$, $x_2(t_0) = k_2, \dots, x_n(t_0) = k_n$.

There is a close relationship between linear differential equations of order n and linear differential systems of the same order. In fact, it turns out that a linear differential equation of order n can always be transformed into a linear system of n differential equations.

To check this statement, consider the equation

$$y^{(n)} + a_1(t)y^{(n-1)} + a_2(t)y^{(n-2)} + \cdots + a_n(t)y = f(t) \tag{2}$$

Finally, the particular solution is given by

$$\boldsymbol{x}_p(t) = \boldsymbol{X}(t) \int \boldsymbol{X}^{-1}(t)\boldsymbol{f}(t)dt \tag{13}$$

Example 3.4

Solve the following system of differential equations:

$$\begin{cases} x_1' = 3x_2 + 30 \\ x_2' = -3x_1 - 3t \end{cases}$$

Answer

In matrix form, the system is written as

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 30 \\ -3t \end{pmatrix}$$

The associated homogeneous system is first solved. So, by calculating the eigenvalues of the coefficient matrix

$$det\begin{pmatrix} 0-\lambda & 3 \\ -3 & 0-\lambda \end{pmatrix} = \begin{vmatrix} -\lambda & 3 \\ -3 & -\lambda \end{vmatrix} = 0$$

it leads to the characteristic equation

$$\lambda^2 + 9 = 0$$

which has as its roots

$$\lambda = \pm 3i$$

Calculating now the eigenvectors, considering the eigenvalue $\lambda = 3i$. The equation to be solved is

$$\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 3i \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

which is equivalent to the system

$$\begin{cases} 3u_2 = 3iu_1 \\ -3u_1 = 3iu_2 \end{cases}$$

CHAPTER 4

LAPLACE TRANSFORM

The Laplace transform is a particularly useful tool in solving linear differential equations with constant coefficients. Its application allows converting an initial-value problem in the variable t into an algebraic problem in the variable s and it is through this that the solution of the differential equation is determined. Problems with discontinuous functions may be considered, such as the Heaviside step function or the Dirac delta "function" (impulse function).

4.1 Definition, existence, and properties of the Laplace transform

For a real variable function $f(t), t \geq 0$ under certain conditions, the Laplace transform is given by $F(s)$ or $\mathcal{L}\{f(t)\}$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)\, e^{-st} dt \qquad (1)$$

as long as the integral exists.

The Laplace transform can be described as $F(s)$ or $\mathcal{L}\{f(t)\}$ or $\mathcal{L}\{f\}$ and is a function of the variable s .

To establish the existence of the Laplace transform, it is necessary to consider certain types of functions. As it will be shown in the next sections, it is possible to determine the Laplace transform for a wide range of functions, even when they have discontinuities.
The next theorem determines under what conditions the Laplace Transform exists.

$$\delta(t-a) = \lim_{k\to 0} f_k(t)$$

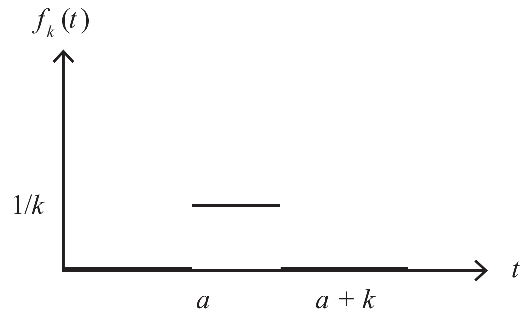


Figure 4.3 Step function $f_k(t)$

This function $f_k(t)$ can be represented using Heaviside step functions as

$$f_k(t) = \frac{1}{k}[u(t-a) - u(t-a-k)]$$

and its Laplace transform is given by

$$\mathcal{L}\{f_k(t)\} = \frac{1}{k}\left(\frac{e^{-as}}{s} - \frac{e^{-(a+k)s}}{s}\right) = e^{-as}\frac{1-e^{-ks}}{ks}$$

Therefore, the Laplace transform of $\delta(t-a)$ is

$$\mathcal{L}\{\delta(t-a)\} = \lim_{k\to 0} e^{-as}\frac{1-e^{-ks}}{ks} = e^{-as}\lim_{k\to 0}\frac{1-e^{-ks}}{ks}$$

Using l´Hopital´s rule

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}\lim_{k\to 0}\frac{se^{-ks}}{s} = e^{-as}\frac{s1}{s}$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}$$

CHAPTER 5

LINE INTEGRAL

The line integral is fundamental in classical mechanics and involves the integration of a function along a domain that is a curve C in $\mathbb{R}^3$, Figure 5.1. The function is either a scalar function or a vector valued function. Curves in $\mathbb{R}^2$ will also be considered for simple examples, as in Example 5.1.

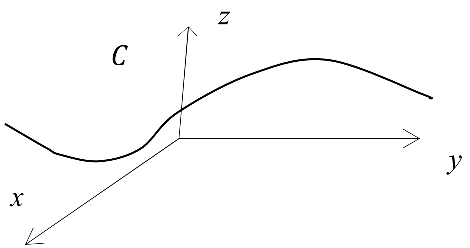


Figure 5.1 Curve in $\mathbb{R}^3$

5.1 Definition of line integral

The line integral is generally presented in the form

$$\int_C \boldsymbol{f} \cdot d\boldsymbol{\alpha} \tag{1}$$

when the function $\boldsymbol{f}$ is a vector field. Next it will be shown how it is evaluated and what is the meaning of this form of line integral.
The curve is usually represented by $\boldsymbol{\alpha}$ where

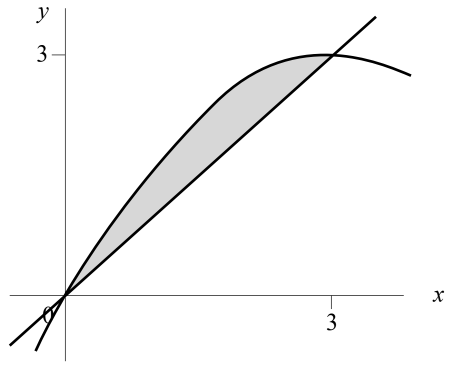


Figure 5.7 Representation of curve C and interior region

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2 - 1 = 1$$

By the theorem, it is obtained that

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dydx = \int_0^3 \int_x^{4x-x^2} 1dydx = \int_0^3 (3x - x^2)\, dx = \frac{9}{2}$$

Now, calculating both line integrals, with the following parameterizations

$\boldsymbol{\alpha}_1(t) = (t, t), t \in [0,3]$, (representing the straight-line segment) and in positive direction
$\boldsymbol{\alpha}_2(t) = (t, 4t - t^2), t \in [0,3]$, (representing the segment of the parabola) and this one, as it is, is traveled in opposite direction (and not upside down), which means that the corresponding integral will now be symmetric.
So, the integral is then given by summing both line integrals in this way

$$\int_0^3 f(\boldsymbol{\alpha}_1(t)) \cdot \boldsymbol{\alpha}_1'(t)dt - \int_0^3 f(\boldsymbol{\alpha}_2(t)) \cdot \boldsymbol{\alpha}_2'(t)dt =$$
$$= \int_0^3 (3 + t, 2t) \cdot (1,1)dt - \int_0^3 (3 + 4t - t^2, 2t) \cdot (1,4 - 2t)dt =$$
$$= \int_0^3 (3 + 3t - 3 - 4t + t^2 - 8t + 4t^2)dt = \frac{9}{2}$$

CHAPTER 6

SURFACE INTEGRAL

A surface S is a two-dimensional exterior, or boundary, of a solid or an object. The surface can be open or closed. A surface is open when it has an edge or boundary, for example a sheet of paper or as the exterior boundary of a cone without including the bottom, just the lateral exterior surface. As an example of a closed surface, it can be mentioned the boundary of a sphere.

A surface can be defined in a unique way, as in the case of a spherical surface, for example, or defined by the six plane faces of a cube, in the case of more than one surface. A surface S is always considered to be without thickness, and if it is bounded the value of its area can be evaluated. In general, it is considered as the geometric space with two degrees of freedom as illustrated in Figure 6.1.

A representation of a surface can be explicit or implicit. For example, a conical surface, represented by the equation

$$z = \sqrt{x^2 + y^2}$$

is explicitly defined. An implicit representation of a surface is in turn of type

$$F(x, y, z) = C \qquad (1)$$

and it could be, in case of the supra mentioned cone $x^2 + y^2 - z^2 = 0, z \geq 0$.

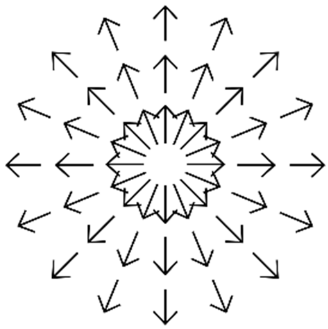
Parametric representation

A parametric representation of a surface, is a vector valued function $\boldsymbol{r}$ defining all its points (x, y, z) with just two parameters:

$$\boldsymbol{r}(u, v) = (x(u, v), y(u, v), z(u, v)), (u, v) \in \Omega \subset \mathbb{R}^2$$

Answer

As it can be verified, by sketching the vectors at some points, this vector field is purely radial, diverges at all points.



$$\nabla \cdot \boldsymbol{F}(x, y, z) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (x, y, z) = 1 + 1 + 1 = 3$$

Example 6.7 Calculate $\nabla \cdot \boldsymbol{F}(x, y, z)$ with $\boldsymbol{F}(x, y, z) = (x^2, 1 + y, -yz)$

Answer

$\nabla \cdot \boldsymbol{F}(x, y, z)$ =$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (x^2,\ 1 + y,\ -yz)$=

=$\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(1 + y) + \frac{\partial}{\partial z}(-yz) =$

$= 2x + 1 - y$

Example 6.8 Curl of a vector field, $rot\boldsymbol{F} = \nabla \times \boldsymbol{F}(x, y, z)$

The curl represents the tendency of the particles in the fluid at points (x, y, z) to rotate around the axis pointing in the direction of $rot\boldsymbol{F}$.

Calculate $\nabla \times \boldsymbol{F}(x, y, z)$ where $\boldsymbol{F}(x, y, z) = (-y, x, 0)$.

Answer

In this case there is no divergence, it is null, and plotting the vectors at several points, they show to describe z-axis centered circles.

Stokes' theorem

Consider S a piecewise smooth, orientable surface with parametric representation $\boldsymbol{r}(T)$ where T is a simple closed plane region, whose boundary is a piecewise smooth Jordan curve C' and $\boldsymbol{r}$ is injective with continuous partial derivatives up to the second order on an open simply connected region. Let C be the image of C' by $\boldsymbol{\alpha}$. If $\boldsymbol{F}(x, y, z)$ is a continuous vector field with continuous partial derivatives then

$$\iint_S curl\boldsymbol{F} \cdot \boldsymbol{n} dS = \oint_C \boldsymbol{F} \cdot d\boldsymbol{\alpha} \qquad (9)$$

$\boldsymbol{n}$ is the unit normal vector to the surface, pointing outward, such that:
For a viewer travelling along C in positive direction having the surface by his left side, the unit normal vector $\boldsymbol{n}$ points upward.

Example 6.12

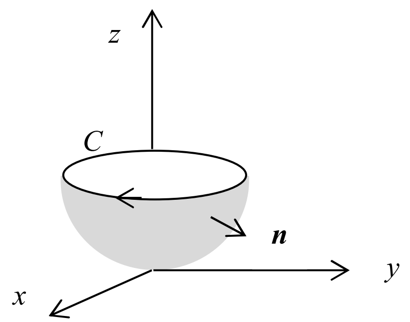
Consider $\boldsymbol{F}(x, y, z) = (y - z^2, -x, -yz)$, use Stokes' theorem to calculate

$$\int\int_S curl\boldsymbol{F} \cdot \boldsymbol{n} dS$$

Where $\boldsymbol{n}$ has positive z-component and is normal to the surface S described by $x^2 + y^2 + z^2 = 2z, z \leq 1$.

Answer

As the flux is inward, line C must be traversed in clockwise direction when viewed from the positive side of the z-axis. Then, the unit normal vector will have negative component in the z -direction.



Calculating $\oint_C \boldsymbol{F} \cdot d\boldsymbol{\alpha}$ where $\boldsymbol{\alpha}(t) = (\cos t, sint, 1)$, with $0 \leq t \leq 2\pi$, as this parametric representation gives counterclockwise direction, multiplication by -1 is needed.

CHAPTER 7

FOURIER SERIES

A Fourier series is an infinite sum of trigonometric functions, sines and cosines, which represents a periodic function. As an example of a trigonometric series, consider the series

$$\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} cos(\,nx) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} sin(\,nx)$$

It can be verified that it is a series containing sine and cosine terms and if the sum exists, that is, if the series converges, it is a periodic function of period 2π.
The purpose of this chapter is to represent periodic functions by combinations of the sine and cosine functions and to obtain trigonometric series called Fourier series. The Fourier series represent a very wide range of functions, even discontinuous. They are allowed to have several points of discontinuity and their application is wide, in various fields of engineering, such as vibration problems, signal theory or differential equations. In the domain of differential equations partial differential equations will be approached with this theory. Wave propagation and heat conduction problems are the most common examples in this field.

In the case of a function having a period 2π, the Fourier series is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n\, cos(nx) + b_n\, sin(\,nx)) \qquad (1)$$

Half-range sine or cosine expansions

In practical problems it often is required to determine a Fourier series for a function $f(x)$ given on the interval $(0,l)$ instead of the full range interval $(-l,l)$. Its odd expansion is such that on the interval $(-l,0)$, $f(x)$ is defined as $f(x) = -f(-x)$. This new function $F(x)$ can then be extended into a periodic function of period $2l$. As $F(x)$ is now odd and periodic, it is represented by a Fourier series called a Fourier sine series and there is no need to calculate the coefficients a_0 a_n. It is sufficient to calculate the coefficient b_n.

Similarly, consider a function $f(x)$ defined on the interval $(0,l)$ (of some physical interest), and consider its even expansion. It is such that on the interval $(-l,0)$ $f(x)$ is defined as $f(x) = f(-x)$. So being an even and periodic function it is represented by a Fourier series called half-range cosine series and there is no need to calculate the coefficient b_n, just a_0 and a_n.

In both these cases using the Dirichlet theorem, $F(x)$ is correctly defined for all x.

Example 7.4

Find the half-range cosine expansion $F(x)$ of $f(x) = x,\ x \in]0,2\pi[$.

Answer

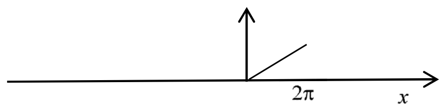


Figure 7.2 Representation of $f(x)$

The plot of the graph, in $\mathbb{R}^2$, of the expansion of the function is given by Figure 7.3

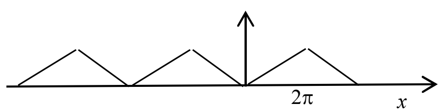
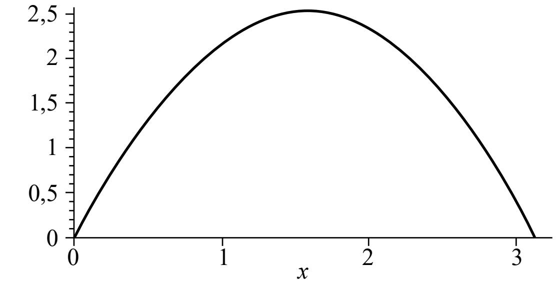
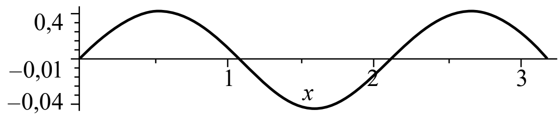


Figure 7.3 $f(x)$ extended as an even periodic function

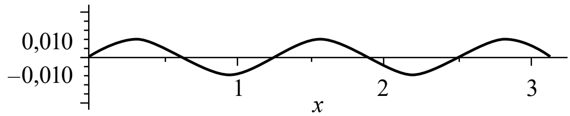
For fixed t, the fundamental mode, for $n = 1$ is given by



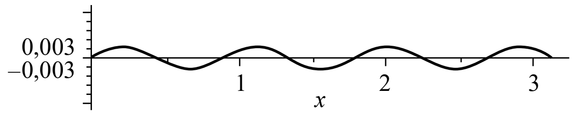
For $n = 3$ is given by



For $n = 5$ is given by



For $n = 7$ is given by



The solution is the superposition of all vibration modes, $n = 1,2,3, \ldots$

LUÍSA MADUREIRA

Problems of Advanced Engineering Mathematics

About the book

This book of problems, designed as a practical tool for engineering students, is based on the author's experience of about 40 years teaching Mathematical Analysis at the University of Porto, Faculty of Engineering. It includes three chapters focused on differential equations, which have wide applications in the field of engineering, as well as a chapter introducing Laplace transforms and its application to ordinary differential equations. The final three chapters cover problems of line and surface integrals, as well as Fourier series.

About the author

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